THE APPROXIMATION OF A PROBLEM OF ANALYTIC DESIGN OF CONTROLS IN A SYSTEM WITH TIME-LAG

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The problem is considered of an optimum control which minimizes the squared error in a system with aftereffect. The approximation of this problem by an analogous problem for systems described by ordinary differential equations, is studied. The convergence of the approximate solution is proved.

1. Let us consider a control system described by Equation

$$\frac{dx}{dt} = Ax(t) + Bx(t-\gamma) + bu \tag{1.1}$$

where x is an n-dimensional phase-coordinate vector, u is a scalar control function, $\gamma > 0$ is a constant time-lag, A, B, b are constant matrices. The motion of system (1.1) is considered in the interval $0 \leqslant t \leqslant T$. The functionals $\eta\,[t, x(\vartheta)]$, defined on the vector-functions

$$x(\vartheta) \quad (-\gamma \leqslant \vartheta \leqslant 0)$$

for $0 \leqslant t \leqslant T$ and such that when $u = \eta\,[t, x(t+\vartheta)]$ Equation (1.1) has the solution $x(t, t_0, z(\vartheta), \eta)$ $(t_0 \leqslant t \leqslant T)$ for any initial conditions t_0 and $x(t_0 + \vartheta) = z(\vartheta)$, where $z(\vartheta)$ is a piecewise-continuous function $(-\gamma \leqslant \vartheta \leqslant 0)$, are called admissible controls. The performance index of (1.1) is evaluated by the functional

$$J\,[t_0, z(\vartheta), u] = \int_{t_0}^{T} \{\omega\,[t, x(t, t_0, z(\vartheta), u)] + u^2(t)\}\, dt + \rho\,[x(T, t_0, z(\vartheta), u)]$$

where $\omega[t,x]$ and $\rho[x]$ are forms with positive terms

$$\omega[t, x] = \sum_{i,\,j=1}^{n} \omega_{ij}(t)\, x_i x_j, \qquad \rho\,[x] = \sum_{i,\,j=1}^{n} \rho_{ij} x_i x_j$$

The analytic design problem of a controller [1] for system (1.1) can be formulated in the following way [2 and 3].

Problem 1.1. From among the admissible controls η to find such a $\eta^\circ[t, x(\vartheta)]$, for which

$$J[t_0, z(\vartheta), \eta^\circ] \leqslant J[t_0, z(\vartheta), \eta]$$

whatever the initial conditions t_0 and $z(\vartheta)$.

This problem has a solution in the form of the linear functional [2 and 3]

$$\eta^\circ[t, x(\vartheta)] = \sum_{i=1}^{n}\left[\alpha_i(t)\, x_i(0) + \int_{-\gamma}^{0} \beta_i(t, \vartheta)\, x_i(\vartheta)\, d\vartheta\right] \qquad (1.2)$$

However, the computation of the functions α_i and β_i is difficult. Therefore, it is advisable to approximate Problem 1.1 by an analogous problem for ordinary differential equations.

Such an approximation was studied in [4] but without investigating the convergence of the approximation. Methods of solving Problem 1.1, based on the approximation of (1.1) by ordinary equations, were worked out and tested by Iu.M. Repin and V.E. Tret'iakov.

In the present article we describe a possible approach to justifying similar approximations.

Let us consider the system of equations (*)

$$\frac{dy^{(i)}}{dt} + my^{(i)} = my^{(i-1)} \qquad \frac{dy^\circ}{dt} = Ay^\circ + By^{(m)} + bu \qquad (i = 1, \ldots, m) \qquad (1.3)$$

where m is an integer, y° and $y^{(i)}$ are n-dimensional vectors, A, B, b are the matrices from (1.1). We shall denote the vector

$$y^\circ, y^{(i)} \ (i = 1, \ldots, m)$$

by the symbol $\{y\}_m$. The functions $\xi_m[t, \{y\}_m]$, defined for $0 \leqslant t \leqslant T$ and such that when $u = \xi_m[t, \{y(t)\}_m]$ the system (1.3) has the solution $\{y(t, t_0, \{z\}_m, \xi_m)\}_m$ for any initial conditions t_0 and $\{y(t_0)\}_m = \{z\}_m$, are called admissible controls (for (1.3)). We denote

$$J_m[t_0, \{z\}_m, u] =$$
$$= \int_{t_0}^{T} \{\omega[t, y^\circ(t, t_0, \{z\}_m, u)] + u^2(t)\}\, dt + \rho[y^\circ(T, t_0, \{z\}_m, u)]$$

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*) System (1.3) is obtained from (1.1) by replacing the term $q(t+\gamma) = x(t)$ which in system (1.1)

$$\frac{dx}{dt} = Ax(t) + Bq(t) + bu$$

effects a shift in the time signal of the amount $\gamma = 1$, by a sequence of m aperiodic terms $dy^{(i)}/dt + my^{(i)} = my^{(i-1)}$, each of which effects a shift in the time signal of approximately the amount $\Delta t = 1/m$ in accordance with the Taylor formula

$$y^{(i-1)}(t) \approx y^{(i)}(t + \Delta t) = y^{(i)}(t) + (dy^{(i)}/dt)\,\Delta t + \ldots$$

P r o b l e m 1.2 . From among the admissible controls ξ_s to find such a $\xi_s^\circ[t,\{y\}_s]$ for which

$$J_m\ [t_0, \{z\}_m, \xi_m{}^\circ] \leqslant J_m\ [t_0, \{z\}_m, \xi_m]$$

whatever the initial conditions t_0 and $\{z\}_s$.

Problem 1.2 has a solution in the form of the linear function [1 and 5]

$$\xi_m{}^\circ\ [t, \{y\}_m] = \sum_{i=1}^{n} \left[\alpha_i{}^{(m)}(t)\, y_i{}^\circ + \sum_{j=1}^{m} \beta_i{}^{(m)}(t, j)\, y_i{}^{(j)}\right] \qquad (1.4)$$

whose coefficients $\alpha_i{}^{(m)}$ and $\beta_i{}^{(m)}$ can be computed by integrating ordinary differential eqautions [6].

The aim of the present paper is to study the relation between the solutions of Problems 1.1 and 1.2 for large m.

2. Let us investigate the relation between the solutions of Equations (1.1) and (1.3) when $u(t) \equiv 0$. We shall assume that $\gamma = 1$. The symbol $\| z \|$ will denote the Euclidean norm of z, i.e. $\| z \| = (z_1{}^2 + \ldots + z_n{}^2)^{1/2}$. The initial function $z(\vartheta)$ is conveniently taken to be an element of the space $L_2[-1, 0]$ with the norm

$$\| z(\vartheta) \|^{(2)}{}_{[-1,0]} = \left[\| z(0) \|^2 + \int_{-1}^{0} \| z(\vartheta) \|^2\, d\vartheta\right]^{1/2}$$

To the motion $x(t)$ of (1.1), resulting from the initial condition $x(t_0+\vartheta) = z(\vartheta)$, let there correspond the motion $\{y(t)\}_m$ of (1.3) with the initial conditions

$$y^\circ(t_0) = z^\circ = z(0), \qquad y^{(i)}(t_0) = z^{(i)} = m \int_{i/m}^{-(i-1)/m} z(\vartheta)\, d\vartheta \qquad (2.1)$$

The solution $x(t)$ of Equation (1.1) satisfies the integral equation

$$x(t) = z(0) + \int_{t_0}^{t} \{Ax(\tau) + \varphi\,[t, \tau]\, Bx(\tau)\}\, d\tau + $$
$$+ \int_{-1}^{0} \varphi^*\,[t_0, t, \vartheta]\, Bz(\vartheta)\, d\vartheta \qquad (2.2)$$

where the functions φ and φ^* are defined by the equalities

$$\begin{aligned} &\varphi\,[t, \tau] = 1 \quad \text{when } \tau < t-1, \quad \varphi\,[t, \tau] = 0 \quad \text{when } t-1 \leqslant \tau \leqslant t \\ &\varphi^*\,[t_0, t, \vartheta] = 1 \quad \text{when } -1 \leqslant \vartheta < \min(0, t-t_0-1) \\ &\varphi^*\,[t_0, t, \vartheta] = 0 \quad \text{when } t-t_0-1 \leqslant \vartheta \leqslant 0 \end{aligned} \qquad (2.3)$$

The solution $y^\circ(t)$ of (1.3) satisfies Equation

$$y^\circ(t) = y^\circ(0) + \int_{t_0}^{t} \{Ay^\circ(\tau) + By^{(m)}(\tau)\}\, d\tau \qquad (2.4)$$

Integrating Equation (1.3) successively we get

$$y^{(m)}(t) = \frac{m^m}{(m-1)!}\int_{t_0}^{t} y^\circ(\tau)(t-\tau)^{m-1}e^{-m(t-\tau)}\,d\tau +$$

$$+\sum_{i=1}^{m} y^{(i)}(0)\frac{[m(t-t_0)]^{m-i}}{(m-i)!}e^{-m(t-t_0)} \tag{2.5}$$

The equation for $y^\circ(t)$ follows from (2.1), (2.4) and (2.5),

$$y^\circ(t) = z(0) + \int_{t_0}^{t}\{Ay^\circ(\tau) + \varphi_m[t,\tau]\,By^\circ(\tau)\}\,d\tau +$$

$$+\int_{-1}^{0}\varphi_m^*[t_0, t, \vartheta]\,Bz(\vartheta)\,d\vartheta \tag{2.6}$$

where the functions φ_m and φ_m^* are defined by the equalities

$$\varphi_m[t,\tau] = \int_{\tau}^{t}\frac{m^{m+1}}{m!}(\zeta-\tau)^{m-1}e^{-m(\zeta-\tau)}\,d\zeta$$

$$\varphi_m^*[t_0, t, \vartheta] = m\int_{0}^{t-t_0}\frac{(m\zeta)^{m-i}}{(m-i)!}e^{-m\zeta}\,d\zeta \tag{2.7}$$

Let us study the function φ_m. We take a small number $\varepsilon > 0$. Replacing $m!$ by the Stirling formula, from (2.7) we get

$$\varphi_m = \frac{\sqrt{m}}{\sqrt{2\pi}}\int_{0}^{t-\tau}[\zeta e^{1-\zeta}]^{m-1}e^{1-\zeta+\theta}\,d\zeta \qquad \left(|\theta| \leqslant \frac{1}{12m}\right)$$

At first let $\tau \geqslant t-1+\varepsilon$. Then $\zeta e^{1-\zeta} \leqslant \lambda < 1$ and, consequently,

$$\lim \varphi_m[t,\tau] = 0 \quad \text{when } m\to\infty \tag{2.8}$$

uniformly in $\tau \geqslant t-1+\varepsilon,\ t\leqslant T$.

Now let $\tau \leqslant t-1-\varepsilon$. Then

$$\varphi_m = \frac{\sqrt{m}}{\sqrt{2\pi}}\left\{\int_{0}^{1-\delta} + \int_{1-\delta}^{1+\delta} + \int_{1+\delta}^{t-\tau}\right\} \qquad (0<\delta<\varepsilon) \tag{2.9}$$

Just as was done previously, we verify that in (2.9) the first and third terms converge to zero as $m\to\infty$, uniformly in τ, $t\leqslant\tau$. We consider the quantity

$$\nu = \frac{\sqrt{m}}{\sqrt{2\pi}}\int_{1-\delta}^{1+\delta} = \frac{\sqrt{m}}{\sqrt{2\pi}}\int_{-\delta}^{\delta}[(1+\zeta)e^{-\zeta}]^{m-1}e^{\theta-\zeta}\,d\zeta = \frac{\sqrt{m}}{\sqrt{2\pi}}\int_{-\delta}^{\delta}\exp\left(-\frac{\xi^2(m-1)}{2+0(\delta)}\right)d\zeta \tag{2.10}$$

Because of the well-known properties of the function $\exp(-\frac{1}{2}\zeta^2)$, from (2.10) it follows that $\nu\to 1+\varkappa(\delta)$ as $m\to\infty$, and $\varkappa(\delta)\to 0$ as $\delta\to 0$. Thus, for any $\varepsilon>0$ there exists $N(\varepsilon)$ such that

$$|\varphi_m[t,\tau]-1| < \varepsilon \quad \text{for } m\geqslant N(\varepsilon),\ \tau\leqslant t-1-\varepsilon$$

uniformly in $t \leqslant T$. Moreover, for all t and τ the function φ_m is uniformly bounded. We study the functions φ_m^* in an analogous way and as a result we arrive at the following conclusion.

Lemma 2.1. The functions φ_m and φ_m^* are uniformly bounded and for any $\varepsilon > 0$ we can find a number $N(\varepsilon)$ such that

$$\operatorname{mes}\left(\left|\varphi_m[t,\tau]-\varphi[t,\tau]\right|>\varepsilon\right)<\varepsilon \qquad (t_0\leqslant\tau\leqslant t) \tag{2.11}$$

$$\operatorname{mes}\left(\left|\varphi_m^*[t,t,\vartheta]-\varphi^*[t_0,t,\vartheta]\right|>\varepsilon\right)<\varepsilon \qquad (-1\leqslant\vartheta\leqslant 0) \tag{2.12}$$

for all $0\leqslant t_0\leqslant t\leqslant T$, if only $m\geqslant N(\varepsilon)$.

Consequently, as $m\to\infty$ the functions φ_m and φ_m^* converge in measure [8] to the functions φ and φ^* on the intervals $[t_0\leqslant\tau\leqslant t]$ and $[-1\leqslant\vartheta\leqslant 0]$ uniformly in t. Further, as a consequence of their uniform boundedness the functions φ_m and φ_m^* converge uniformly in-the-mean to the functions φ and φ^* on the same intervals [8].

From Equations (2.2) and (2.6), from the properties of the functions φ, φ^*, φ_m and φ_m^*, it follows that the motions $x(t)$ and $y^\circ(t)$, corresponding to the initial conditions

$$\|z(\vartheta)\|^{(2)}{}_{[-1,0]}\leqslant 1 \tag{2.13}$$

are uniformly unbounded. The difference $f(t)=y^\circ(t)-x(t)$, according to (2.2) and (2.6), satisfies Equation

$$f(t)=\int_{t_0}^{t}\{A+\varphi_m[t,\tau]B\}f(\tau)\,d\tau+\int_{t_0}^{t}\{\varphi_m[t,\tau]-\varphi[t,\tau]\}Bx(\tau)\,d\tau+$$
$$+\int_{-1}^{0}\varphi_m^*[t_0,t,\vartheta]-\varphi^*[t_0,t,\vartheta]\}Bz(\vartheta)\,d\vartheta \tag{2.14}$$

We shall consider $f(t)$ to be an element in the space $L_2[t_0,T]$ with the norm

$$\|f(t)\|^{(2)}{}_{[t_0,T)}=\left[\int_{t_0}^{T}\|f(t)\|^2\,dt\right] \tag{2.15}$$

The operator

$$H[f]=f(t)-\int_{t_0}^{t}\{A+\varphi_m B\}f(\tau)\,d\tau \tag{2.16}$$

has an inverse [9 and 10] H^{-1} which is uniformly bounded in $t_0\leqslant T$ and in m. From (2.14) it follows that

$$f(t)=H^{-1}\left[\int_{t_0}^{t}\{\varphi_m-\varphi\}Bx(\tau)\,d\tau+\int_{-1}^{0}\{\varphi_m^*-\varphi^*\}Bz(\vartheta)\,d\vartheta\cdot\right] \tag{2.17}$$

and from the properties of the functions φ_m, φ, φ_m^*, φ^*, (2.11), (2.12), $z(\vartheta)$ of (2.13) and $x(t)$, we conclude that

$$\lim\|f(t)\|^{(2)}{}_{[t_0,T)}=0 \quad \text{when } m\to\infty \tag{2.18}$$

uniformly in $t_0 \leqslant T$ and in the $z(\vartheta)$ of (2.13). From (2.14) and (2.18) it follows that

$$\lim \| f(t) \|^c_{[t_0, T]} = 0 \quad \text{when } m \to \infty \tag{2.19}$$

uniformly in $t_0 \leqslant T$ and in the $z(\vartheta)$ of (2.13). Here

$$\| f(t) \|^c_{[t_0, T]} = \max (\| f(t) \| \quad \text{when } t_0 \leqslant t \leqslant T) \tag{2.20}$$

By the symbols $h_{ij}[t, t_0]$ and $h_{ij}^{(m)}[t, t_0]$ we denote the solutions $x_i(t)$ and $y_i^\circ(t)$ of systems (1.1) and (1.3) generated when $u = 0$ by the initial conditions $z(\vartheta, j)$, where

$$z_j(0, j) = 1, \quad z_k(0, j) = 0, \quad z(\vartheta, j) = 0 \quad (k \neq j, -1 \leqslant \vartheta < 0)$$

The symbols $h_{(\delta)}[t, t_0]$ and $h_{(\delta)}^{(m)}[t, t_0]$ denote the impulse responses of systems (1.1) and (1.3), i.e. the motions of these systems generated at $t = t_0$ by zero initial conditions and by the control $u(t) = \delta(t - t_0)$, where δ is the delta-function. We have

$$h_{(\delta)i} = \sum_{j=1}^{n} h_{ij} b_j(t_0), \qquad h_{(\delta)i}^{(m)} = \sum_{j=1}^{n} h_{ij}^{(m)} b_j(t_0)$$

From (2.20) we arrive at the following conclusion.

Lemma 2.2. For any $\varepsilon > 0$ we can find a number N_ε such that

$$\| y^\circ(t) - x(t) \|^c_{[t_0, T]} < \varepsilon \tag{2.21}$$

$$\| h_j[t, t_0] - h_j^{(m)}[t, t_0] \|^c_{[t_0, T]} < \varepsilon \tag{2.22}$$

for all $t_0 \leqslant T$ and for the $z(\vartheta)$ of (2.13), if only $m \geqslant N_\varepsilon$.

Note 2.1. The integrals

$$\int_{-1}^{0} \varphi^* B z(\vartheta)\, d\vartheta, \qquad \int_{-1}^{0} \varphi_m^* B z(\vartheta)\, d\vartheta$$

in Equations (2.2) and (2.6) can be treated as the mathematical expectations of certain random variables $\varkappa[\vartheta, \tau]$ and $\varkappa_m[\vartheta, \tau]$, generated by the function $z(\vartheta)$ and having a Poisson distribution [11]; moreover, $\varkappa = \varkappa_m = Bz(\vartheta)$ and the probability density $p_m[\vartheta, \tau]$ is defined by the equality

$$p_m[\vartheta, \tau] = \frac{m}{(m - i)!}(m\tau)^{m-i} e^{-m\tau} \quad \text{when } -\frac{i}{m} \leqslant \vartheta < -\frac{i-1}{m}; \quad \tau \leqslant t,$$

($p_m[\vartheta, \tau] = 0$ for other ϑ and τ), and the density $p[\vartheta, \tau]$ will be the limit of p_m as $m \to \infty$.

The integrals

$$\int_{t_0}^{t} \varphi B x\, d\tau, \qquad \int_{t_0}^{t} \varphi_m B y^\circ\, d\tau$$

can be treated analogously.

It is therefore natural that the convergence studied in Section 2 analogously corresponds to limit relations in probability theory [11].

3. Let us consider the relation between the optumum controls η° and ξ_m° of Problems 1.1 and 1.2. The optimum control for Problem 1.1 under the initial condition $x(t_0 + \vartheta) = z(\vartheta)$, which can be considered as a function of time, will be denoted by the symbol $u^\circ(t, t_0, z(\vartheta))$.

The analogous control for Problem 1.2 under the initial conditions $\{y(t_0)\}_m$ of (2.1) is denoted by $u_m^\circ(t, t_0, \{z\}_m) = u_m^\circ(t, t_0, z(\vartheta))$. Moreover, we denote

$$v[t, z(\vartheta)] = J[t, z(\vartheta), u^\circ]$$

$$v_m[t, z(\vartheta)] = v_m[t, \{z\}_m] = J_m[t, z(\vartheta), u_m^\circ]$$

The functionals v and v_m satisfy the equation of R.Bellman [12], and hence it follows that [1 to 3]

$$\begin{aligned} u^\circ(t, t_0, z(\vartheta)) &= -\frac{1}{2}\sum_{i=1}^{n} b_i \frac{\partial v[t, x(t+\vartheta)]}{\partial x_i(t)} = -\frac{1}{2} b\psi \\ u_m^\circ(t, t_0, \{z\}_m) &= -\frac{1}{2}\sum_{i=1}^{n} b_i \frac{\partial v_m[t, \{y(t)\}_m]}{\partial y_i^\circ(t)} = -\frac{1}{2} b\psi_m \end{aligned} \tag{3.1}$$

Here the vectors $\{\partial v / \partial x_i\}$ and $\{\partial v_m\}/\partial y_i^\circ\}$ have the meaning of the vector ψ from the maximum principle of Pontriagin [13]. According to (3.1)

$$\begin{aligned} u^\circ(t, t_0, z(\vartheta)) = -\sum_{i=1}^{n} b_i \Big[\int_t^T \Big\{\sum_{j,\, l=1}^{n} \omega_{jl} x_j^*(\tau) \frac{\partial x_l^*(\tau)}{\partial x_i(t)}\Big\} d\tau + \\ + \sum_{j,\, l=1}^{n} \rho_{jl} x_j^*(T) \frac{\partial x_l^*(T)}{\partial x_i(t)}\Big] \end{aligned} \tag{3.2}$$

$$\begin{aligned} u_m^\circ(t, t_0, z(\vartheta)) = -\sum_{i=1}^{n} b_i \Big[\int_t^T \Big\{\sum_{j,\, l=1}^{n} \omega_{jl} y_j^{\circ *}(\tau) \frac{\partial y_l^{\circ *}(\tau)}{\partial y_i^\circ(t)}\Big\} d\tau + \\ + \sum_{j,\, l=1}^{n} \rho_{jl} y_j^{\circ *}(T) \frac{\partial y_l^{\circ *}(T)}{\partial y_i^\circ(t)}\Big] \end{aligned} \tag{3.3}$$

where x^* and $y^{\circ *}$ are the optimum motions corresonding to the initial conditions $x(t_0 + \vartheta) = z(\vartheta)$ and $\{y(t_0)\}_m$ of (2.1); the coordinates $x_l^*(\tau)$ and $y_l^{\circ *}(\tau)$ are differentiated with respect to the coordinates $x_i(t)$ and $y_i^\circ(t)$ for fixed u° and u_m° since as a consequence of the optimum of u° and u_m° the variations of the functionals $v = J$ and $v_m = J_m$ brought about by the variations of the controls, equal zero. The optimum motions are determined by the Cauchy formula [14]

$$\begin{aligned} x^*(t) &= x(t) + \int_{t_0}^{t} h_{(\delta)}[t, \tau]\, u^\circ(\tau, t_0, z(\vartheta))\, d\tau \\ y^{\circ *}(t) &= y^\circ(t) + \int_{t_0}^{t} h_{(\delta)^{(m)}}[t, \tau]\, u_m^\circ(\tau, t_0, z(\vartheta))\, d\tau \end{aligned} \tag{3.4}$$

where $x(t)$ and $y^\circ(t)$ are the solutions of the corresponding homogeneous equations. Moreover,

$$\frac{\partial x_l(\tau)}{\partial x_i(t)} = h_{li}[\tau, t], \qquad \frac{\partial y_l^\circ(\tau)}{\partial y_i^\circ(t)} = h_{li}^*[\tau, t] \tag{3.5}$$

From (3.2) to (3.5) follow the integral equations for u° and $u_m{}^\circ$

$$u^\circ(t) = -\sum_{i=1}^{n} b_i \Big[\int_t^T \Big\{\sum_{j,\,l=1}^{m} \omega_{jl} h_{li}[\tau, t] \Big(\int_{t_0}^{\tau} h_{(\delta)j}[\tau, \zeta]\, u^\circ(\zeta)\, d\zeta + x_j(\tau)\Big)\Big\} d\tau + \sum_{j,\,l=1}^{n} \rho_{jl} h_{li}[T, t] \Big(\int_{t_0}^{T} h_{(\delta)j}[T, \zeta]\, u^\circ(\zeta)\, d\zeta + x_j(T)\Big)\Big] \tag{3.6}$$

$$u_m{}^\circ(t) = -\sum_{i=1}^{n} b_i \Big[\int_t^T \Big\{\sum_{j,\,l=1}^{n} \omega_{jl} h_{li}{}^{(m)}[\tau, t] \Big(\int_{t_0}^{\tau} h_{(\delta)j}{}^{(m)}[\tau, \zeta]\, u_m{}^\circ(\zeta)\, d\zeta + y_j{}^\circ(\tau)\Big)\Big\} d\tau + \sum_{j,\,l=1}^{n} \rho_{jl} h_{li}{}^{(m)}[T, t] \Big(\int_{t_0}^{T} h_{(\delta)j}{}^{(m)}[T, \zeta]\, u^\circ(\zeta)\, d\zeta + y_j^\circ(T)\Big)\Big] \tag{3.7}$$

The operator $G[u]$ corresponding to Equation (3.6) is

$$G[u] = u(t) + \sum_{i=1}^{n} b_i \sum_{j,l=1}^{n} \Big\{\int_t^T \omega_{jl} h_{li}[\tau, t] \Big(\int_{t_0}^{\tau} h_{(\delta)j}[\tau, \zeta]\, u(\zeta)\, d\zeta\Big) d\tau + \rho_{jl} h_{li}[T, t] \int_{t_0}^{T} h_{(\delta)j}[T, \zeta]\, u(\zeta)\, d\zeta\Big\} \tag{3.8}$$

and has the inverse G^{-1} uniformly bounded in $t_0 \leqslant T$, when considered to be in the space $L_2[t_0, T]$. Indeed, let $G[u] = f(t)$.

From (3.8) it follows that

$$\int_{t_0}^{T} f(t)\, u(t)\, dt = \int_{t_0}^{T} u^2(t)\, dt + \int_{t_0}^{T} \sum_{j,\,l=1}^{n} \omega_{jl}(\tau) \Big[\int_{t_0}^{\tau} h_{(\delta)l}[\tau, t]\, u(t)\, dt\Big]\Big[\int_{t_0}^{\tau} h_{(\delta)j}[\tau, \zeta]\, u(\zeta)\, d\zeta\Big] d\tau + \sum_{j,\,l=1}^{n} \rho_{jl} \Big(\int_{t_0}^{T} h_{(\delta)l}[T, t]\, u(t)\, dt\Big)\Big(\int_{t_0}^{T} h_{(\delta)j}[T, \zeta]\, u(\zeta)\, d\zeta\Big) \geqslant \int_{t_0}^{T} u^2(t)\, dt \tag{3.9}$$

as a consequence of the fact that $\omega[t, x]$ and $\rho[x]$ have positive signs. Therefore, the norm

$$\| f \|^{(2)}{}_{[t_0,\, T)} \geqslant \| u \|^{(2)}{}_{[t_0,\, T)}$$

whence the existence and boundedness of G^{-1} follows [9 and 10]. An analogous conclusion is valid for the operator G_m corresponding to (3.7). From the properties of operators G^{-1} and G_m^{-1}, and also from the properties of the motions $x(t)$ and $y^\circ(t)$ and of the functions h_j and $h_j{}^{(m)}$, noted in Lemma 2.2, we conclude that the following assertion is valid.

T h e o r e m 3.1 . The optimum controls $u^\circ(t, t_0, z(\vartheta))$ and $u_m{}^\circ(t, t_0, \{z\}_m)$, corresponding to the initial conditions (2.13) and (2.1), are uniformly bounded.

For any $\epsilon > 0$ we can find a number N_ϵ such that

$$| u^\circ (t, t_0, z(\vartheta)) - u_m{}^\circ (t, t_0, z(\vartheta)) | < \varepsilon \qquad (\| z(\vartheta) \|^{(2)}{}_{[-1,0]} \leqslant 1) \quad (3.10)$$

for all $t_0 \leqslant T$, if only $m \geqslant N_\varepsilon$.

From the convergence (2.21) of the motion $y^\circ(t)$ to $x(t)$, from (2.22), and from the convergence (3.10) of controls u_m° to u°, we conclude the following lemma.

Lemma 3.1. The optimum motions $y^{\circ *} (t, t_0, \{z\}_m)$ converge uniformly to the motions $x^* (t, t_0, z(\vartheta))$ for all $t_0 < T$ and for all initial curves $z(\vartheta)$ of (2.13) and $\{z\}_m$ of (2.1).

The equalities

$$\eta^\circ [t, z(\vartheta)] = u^\circ (t, t, z(\vartheta)), \qquad \xi_m{}^\circ [t, z(\vartheta)] = u_m{}^\circ (t, t, z(\vartheta))$$

are valid by the definitions of u° and $u_m{}^\circ$ and of η° and $\xi_m{}^\circ$.

The following assertion is a consequence of Theorem 3.1 and Lemma 3.1

Theorem 3.2. For any $\epsilon > 0$ we can find a number N_ϵ such that

$$| \eta^\circ [t, z(\vartheta)] - \xi_m{}^\circ [t, \{z\}_m] | \leqslant \varepsilon \| z(\vartheta) \|^{(2)}{}_{[-1, 0]} \qquad (3.11)$$

$$| J [t, z(\vartheta), \eta^\circ] - J_m [t, \{z\}_m, \xi_m{}^\circ] | \leqslant \varepsilon (\| z(\vartheta) \|^{(2)}{}_{[-1, 0]})^2 \qquad (3.12)$$

for all $t \in [0, T]$ and for the $\{z\}_m$ of (2.1), if only $m \geqslant N_\varepsilon$.

4. The theorems of Section 3 establish the specific convergence of the solutions of the auxiliary problem to the solutions of the original problem. However, there still remains unanswered here the fundamental question: whether or not the motions of system (1.1) which are generated in the plant (1.1) by the control law found from the solutions of the auxiliary problem will be close to the optimum motions? Let us discuss this question here.

Let the initial curve $z(\vartheta)$ $(-1 \leqslant \vartheta \leqslant 0)$ be chosen from any compact set of functions $z(\vartheta)$. For definiteness we shall assume, for example, that the initial state $z(\vartheta)$ is chosen from among piecewise-continuous functions, uniformly bounded thus $\| z(\vartheta) \| \leqslant 1$, having not more than one point of discontinuity. We shall assume that on the continuous segments the function $z(\vartheta)$ is equicontinuous. The optimum motions $x(t)$ and $y^\circ(t)$ of systems (1.1) and (1.3), generated by these initial states, are uniformly bounded and equicontinuous. By the symbol $w_m(t)$ we denote the motion of system (1.1) when $u = \xi_m[t, \{w_m(t)\}_m]$, where the vector $\{w_m(t)\}_m = \{w_m(t), w_m(t-(1/m)), \ldots, w_m(t-1)\}$. In other words, $w_m(t)$ is the motion of system (1.1) which is obtained if to the plant (1.1) with aftereffect is applied the control law found for the auxiliary Problem 1.2. The motions $w_m(t)$ also will be uniformly bounded and equicontinuous under bounded initial states from the above-mentioned set of functions $z(\vartheta)$, which we denote by $[z(\vartheta)]_{(k)}$. But in such a case, from the estimates of Section 3 the following assertion is valid.

T h e o r e m 4.1. For any number $\varepsilon > 0$ there exists a number N_ε such that

$$\| x (t, t_0, z(\vartheta), \eta^\circ) - w_m (t, t_0, z(\vartheta), \xi_m^\circ) \| < \varepsilon \quad \text{when } t \geqslant t_0 \tag{4.1}$$

for all t_0, $z(\vartheta) \in [z(\vartheta)]_{(k)}$ and $m \geqslant N_\varepsilon$.

An analogous conclusion of convergence is valid also for the values of the quantity J to be minimized.

N o t e 4.1. The bounds on the class of initial curves $z(\vartheta)$ from the compact set $[z(\vartheta)]_{(k)}$ are not essential for the validity of (4.1) since the uniform convergence $w_m(t) \to x(t)$ is preserved, for example, also for all initial conditions $\|z(\vartheta)\| \leqslant 1$ where $z(\vartheta) \in L_2[-1, 0]$; conversely, the uniform convergence of the quantity J being minimized cannot be obtained under such an extension of the class of initial states $z(\vartheta)$. The proof and the analysis of the assertions made in this Note are outside the scope of the present paper.

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